PERIODICITY OF TWO FUZZY DIFFERENCE EQUATIONS

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ABSTRACT: In this paper we study the periodicity of the solutions of the following fuzzy difference equations

$$x_{n+1} = \max\Big\{\frac{A}{x_n}, \frac{A}{x_{n-1}}, ..., \frac{A}{x_{n-k}}\Big\}, \quad x_{n+1} = \max\Big\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}\Big\}$$

where k is a positive integer, A, A_i, $i = 0, 1$ are positive real constants and the initial values x_i, $i = -k, -k+1, ..., 0$ (resp. $i = -1, 0$) of the first (resp. second) equation are positive fuzzy numbers.

Key words: Fuzzy difference equations, eventually periodic, fuzzy number, a-cuts.
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1 INTRODUCTION

Difference Equations are often used in the study of linear and nonlinear physical, physiological and economical problems (for partial review see [3], [6]). This fact leads to the fast promotion of the theory of Difference Equations which someone can find for instance in [1], [7], [9].

Nowadays, a modern and promising approach for engineering, social and environmental problems with imprecise, uncertain input-output data arises, the fuzzy approach. This is an expectable effect, since Fuzzy Logic can handle various types of vagueness but particulary vagueness related to human linguistic and thinking (for partial review see [8], [10]).

The increasing interest in applications of these two scientific fields contributed the appearance of Fuzzy Difference Equations (see [4], [5], [11], [12], [13]).

*This work ia a part of her Doctoral Thesis

In this paper we study the fuzzy difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{A}{x_{n-1}}, ..., \frac{A}{x_{n-k}}\right\} \tag{1.1}$$

where k is a positive integer, A is a positive real constant, x_i, $i = -k, -k+1, ..., 0$ are positive fuzzy numbers and the fuzzy difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}\right\} \tag{1.2}$$

where A_0, A_1 are positive real constants and x_{-1}, x_0 are positive fuzzy numbers. Firstly, we prove that for every positive fuzzy numbers $x_{-k}, x_{-k+1}, ..., x_0$ (resp. x_{-1}, x_0) there exists a unique solution x_n of (1.1) (resp. (1.2)) with initial values $x_{-k}, x_{-k+1}, ..., x_0$ (resp. x_{-1}, x_0). Moreover, we prove that every positive solution of (1.1) is eventually periodic of period $k+2$. Finally, we prove that every positive solution of (1.2) is eventually periodic of period 2 if $A_0 > A_1$, 3 if $A_1 = A_0$ and 4 if $A_0 < A_1$.

We note that the results obtained in this paper extend the analogous results obtained by Szalkai in [14] (resp. by Amleh, Hoag and Ladas in [2]) where the author (resp. authors) studied the corresponding ordinary difference equation (1.1) (resp. (1.2)) where k is a positive integer, A, A_0, A_1 are positive real constants and x_i, $i = -k, -k+1, ..., 0$, (resp. x_{-1}, x_0) are positive real numbers.

2 PRELIMINARIES

We need the following definitions:

For a set B we denote by $\overline{B}$ the closure of B. We say that a function A from $I\!R^+ = (0, \infty)$ into the interval $[0, 1]$ is a fuzzy number if A is normal, convex fuzzy set (see [11]), upper semicontinuous and the support $\text{supp}A = \overline{\bigcup_{a\in(0,1]} [A]_a} = \overline{\{x : A(x) > 0\}}$ is compact. Then from Theorems 3.1.5 and 3.1.8 of [10] the a-cuts of the fuzzy number A, $[A]_a = \{x \in I\!R^+ : A(x) \geq a\}$ are closed intervals.

We say that a fuzzy number A is positive if $\text{supp}A \subset (0, \infty)$.

It is obvious that if A is a positive real number then A is a positive fuzzy number and $[A]_a = [A, A]$, $a \in (0, 1]$. In this case, we say that A is a trivial fuzzy number.

Let B_i, $i = 0, 1, ..., k$, k is a positive integer be fuzzy numbers such that

$$[B_i]_a = [B_{i,l,a}, B_{i,r,a}], \quad i = 0, 1, ..., k, \quad a \in (0, 1].$$

Let for any $a \in (0, 1]$

$$C_{l,a} = \max\{B_{i,l,a}, \ i = 0, 1, ..., k\}, \quad C_{r,a} = \max\{B_{i,r,a}, \ i = 0, 1, ..., k\}.$$

Then by Theorem 2.1 of [15] $(C_{l,a}, C_{r,a})$ determines a fuzzy number C such that

$$[C]_a = [C_{l,a}, C_{r,a}], \quad a \in (0,1].$$

According to [8] and Lemma 2.3 of [12] we can define

$$C = \max\{B_i, \ i = 0, 1, ..., k\}.$$

We say that x_n is a positive solution of (1.1) (resp. (1.2)) if x_n is a sequence of positive fuzzy numbers which satisfies (1.1) (resp. (1.2)).

A solution x_n of (1.1) (resp.(1.2)) is said to be eventually periodic of period r, r is a positive integer, if there exists a positive integer m such that

$$x_{n+r} = x_n, \quad n = m, m+1, ...$$

3 EXISTENCE AND UNIQUENESS OF THE POSITIVE SOLUTIONS OF FUZZY DIFFERENCE EQUATIONS (1.1) AND (1.2)

In this section we study the existence and the uniqueness of the positive solutions of the fuzzy difference equations (1.1) and (1.2).

Proposition 3.1 *For every positive fuzzy numbers $x_{-k}, x_{-k+1}, ..., x_0$ (resp. x_{-1}, x_0) there exists a unique positive solution x_n of (1.1) (resp. (1.2)) with initial values $x_{-k}, x_{-k+1}, ..., x_0$ (resp. x_{-1}, x_0).*

Proof Let x_i, $i = -k, -k+1, ..., 0$ be positive fuzzy numbers such that

$$[x_i]_a = [L_{i,a}, R_{i,a}], \quad i = -k, -k+1, ..., 0, \quad a \in (0,1]. \tag{3.1}$$

Let $(L_{n,a}, R_{n,a})$, $n = 0, 1, ...$, $a \in (0,1]$ be the unique positive solution of the system of difference equations

$$\begin{aligned} L_{n+1,a} &= \max\left\{\frac{A}{R_{n,a}}, \frac{A}{R_{n-1,a}}, ..., \frac{A}{R_{n-k,a}}\right\}, \\ R_{n+1,a} &= \max\left\{\frac{A}{L_{n,a}}, \frac{A}{L_{n-1,a}}, ..., \frac{A}{L_{n-k,a}}\right\} \end{aligned} \tag{3.2}$$

with initial values $(L_{i,a}, R_{i,a})$, $i = -k, -k+1, ..., 0$. Since $A, L_{i,a}, R_{i,a}$, $i = -k, -k+1, ..., 0$, $a \in (0,1]$ are positive numbers, then using Theorem 2.1 of [15] and working as in Proposition 2.1 of [11] and Proposition 1 of [13] we can easily prove that $[L_{n,a}, R_{n,a}]$, $n = 1, 2, ...,$ $a \in (0, 1]$ determines a sequence of fuzzy numbers x_n such that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \ n = 1, 2, ..., \ a \in (0, 1]. \tag{3.3}$$

Now, we prove that x_n satisfies (1.1) with initial values x_i, $i = -k, -k+1, ..., 0$. From (3.1), (3.2), (3.3), Lemma 1 of [13] and by a slight generalization of Lemma 2.3 of [12] we have

$$[\max\{\frac{A}{x_n}, \frac{A}{x_{n-1}}, ..., \frac{A}{x_{n-k}}\}]_a =$$

$$[\max\{\frac{A}{R_{n,a}}, \frac{A}{R_{n-1,a}}, ..., \frac{A}{R_{n-k,a}}\}, \max\{\frac{A}{L_{n,a}}, \frac{A}{L_{n-1,a}}, ..., \frac{A}{L_{n-k,a}}\}] =$$

$$[L_{n+1,a}, R_{n+1,a}] = [x_{n+1}]_a, \ a \in (0, 1]. \tag{3.4}$$

From (3.4) and arguing as in Proposition 2.1 of [11] and Proposition 1 of [13] we have that x_n is the unique positive solution of (1.1) with initial values x_i, $i = -k, -k+1, ..., 0$.

Arguing as above we can easily prove that if x_i, $i = -1, 0$ are positive fuzzy numbers which satisfy

$$[x_i]_a = [L_{i,a}, R_{i,a}], \ i = -1, 0, \ a \in (0, 1] \tag{3.5}$$

then there exists a unique positive solution x_n of (1.2) with initial values x_i, $i = -1, 0$ such that $[x_n]_a = [L_{n,a}, R_{n,a}]$, $n = 1, 2, ...,$ $a \in (0, 1]$ and $(L_{n,a}, R_{n,a})$ satisfies the system of difference equations

$$L_{n+1,a} = \max\{\frac{A_0}{R_{n,a}}, \frac{A_1}{R_{n-1,a}}\}, \quad R_{n+1,a} = \max\{\frac{A_0}{L_{n,a}}, \frac{A_1}{L_{n-1,a}}\}. \tag{3.6}$$

This completes the proof of the proposition.

4 PERIODICITY OF THE SOLUTIONS OF FUZZY EQUATION (1.1)

In this section we study the periodicity of the positive solutions of (1.1). We need the following lemmas.

Lemma 4.1 *Let A, a, b be positive numbers such that $ab \neq A$. If*

$$ab < A \quad (\textit{resp. } ab > A) \tag{4.1}$$

then there exist positive numbers $\bar{y}, \bar{z}$ such that

$$\bar{y}\bar{z} = A \tag{4.2}$$

and

$$a < \bar{y}, \quad b < \bar{z} \quad (\textit{resp. } a > \bar{y}, \quad b > \bar{z}). \tag{4.3}$$

Proof Suppose that (4.1) is satisfied. Then if ϵ is a positive number such that

$$\epsilon < \frac{A - ab}{b} \quad (\text{resp. } \epsilon < \frac{ab - A}{b})$$

and

$$\bar{y} = a + \epsilon, \quad \bar{z} = \frac{A}{a+\epsilon} \quad (\text{resp. } \bar{y} = a - \epsilon, \quad \bar{z} = \frac{A}{a-\epsilon})$$

it is obvious that (4.2) and (4.3) hold. This completes the proof of the lemma.

Lemma 4.2 *We consider the system of difference equations*

$$y_{n+1} = \max\Big\{\frac{A}{z_n}, \frac{A}{z_{n-1}}, ..., \frac{A}{z_{n-k}}\Big\}, \quad z_{n+1} = \max\Big\{\frac{A}{y_n}, \frac{A}{y_{n-1}}, ..., \frac{A}{y_{n-k}}\Big\} \tag{4.4}$$

where A is a positive real constant, k is a positive integer and y_i, z_i, $i = -k, -k+1, ..., 0$ are positive real numbers. Then every positive solution (y_n, z_n) of (4.4) is eventually periodic of period $k+2$.

Proof Let (y_n, z_n) be an arbitrary positive solution of (4.4). Firstly, suppose that there exists a $\lambda \in \{1, 2, ..., k+2\}$ such that

$$y_\lambda z_\lambda < A. \tag{4.5}$$

Then from (4.5) and Lemma 4.1 there exist positive constants $\overline{y}, \overline{z}$ such that (4.2) holds and

$$y_\lambda < \overline{y}, \quad z_\lambda < \overline{z}. \tag{4.6}$$

From (4.2), (4.4) and (4.6) we have for $i = \lambda+1, \lambda+2, ..., k+\lambda+1$

$$\begin{aligned} y_i &= \max\Big\{\frac{A}{z_{i-1}}, \frac{A}{z_{i-2}}, ..., \frac{A}{z_{i-k-1}}\Big\} \ge \frac{A}{z_\lambda} > \frac{A}{\overline{z}} = \overline{y}, \\ z_i &= \max\Big\{\frac{A}{y_{i-1}}, \frac{A}{y_{i-2}}, ..., \frac{A}{y_{i-k-1}}\Big\} \ge \frac{A}{y_\lambda} > \frac{A}{\overline{y}} = \overline{z}. \end{aligned} \tag{4.7}$$

Then relations (4.2), (4.4) and (4.7) imply that

$$\begin{aligned} y_{k+\lambda+2} &= \max\Big\{\frac{A}{z_{k+\lambda+1}}, \frac{A}{z_{k+\lambda}}, ..., \frac{A}{z_{\lambda+1}}\Big\} < \frac{A}{\overline{z}} = \overline{y} \\ z_{k+\lambda+2} &= \max\Big\{\frac{A}{y_{k+\lambda+1}}, \frac{A}{y_{k+\lambda}}, ..., \frac{A}{y_{\lambda+1}}\Big\} < \frac{A}{\overline{y}} = \overline{z}. \end{aligned} \tag{4.8}$$

Therefore, from (4.4) and (4.8) we take for $j = k+\lambda+3, k+\lambda+4, ..., 2k+\lambda+3$

$$\begin{aligned} y_j &= \max\Big\{\frac{A}{z_{j-1}}, \frac{A}{z_{j-2}}, ..., \frac{A}{z_{j-k-1}}\Big\} = \frac{A}{z_{k+\lambda+2}}, \\ z_j &= \max\Big\{\frac{A}{y_{j-1}}, \frac{A}{y_{j-2}}, ..., \frac{A}{y_{j-k-1}}\Big\} = \frac{A}{y_{k+\lambda+2}}. \end{aligned} \tag{4.9}$$

So, from (4.4), (4.8), (4.9) and working inductively for $i = 0, 1, ...$ and $j = 3, 4, ..., k+3$ we can easily prove that

$$y_{k+\lambda+2+i(k+2)} = y_{k+\lambda+2}, \; y_{k+\lambda+j+i(k+2)} = \frac{A}{z_{k+\lambda+2}},$$
$$z_{k+\lambda+2+i(k+2)} = z_{k+\lambda+2}, \; z_{k+\lambda+j+i(k+2)} = \frac{A}{y_{k+\lambda+2}}$$

and so it is obvious that (y_n, z_n) is eventually periodic of period $k+2$.

Therefore, if relation

$$y_{k+2}z_{k+2} < A$$

holds then (y_n, z_n) is eventually periodic of period $k+2$.

Now, suppose that relation

$$y_{k+2}z_{k+2} > A \tag{4.10}$$

is satisfied. Then from (4.10) and Lemma 4.1 there exist positive constants $\overline{y}, \overline{z}$ such that (4.2) holds and

$$y_{k+2} > \overline{y}, \quad z_{k+2} > \overline{z}. \tag{4.11}$$

Moreover, from (4.4) and (4.11) there exist $\lambda, \mu \in \{1, 2, ..., k+1\}$ such that

$$\begin{aligned} y_{k+2} &= \max\Big\{\frac{A}{z_{k+1}}, \frac{A}{z_k}, ..., \frac{A}{z_1}\Big\} = \frac{A}{z_\lambda} > \overline{y} \\ z_{k+2} &= \max\Big\{\frac{A}{y_{k+1}}, \frac{A}{y_k}, ..., \frac{A}{y_1}\Big\} = \frac{A}{y_\mu} > \overline{z} \end{aligned} \tag{4.12}$$

Hence, from (4.2) and (4.12) it follows that

$$z_\lambda < \overline{z}, \quad y_\mu < \overline{y}. \tag{4.13}$$

We prove that $\lambda = \mu$. Suppose on the contrary that $\lambda \neq \mu$. Without loss of generality we may suppose that $1 \leq \mu \leq \lambda - 1$. Then from (4.2), (4.4) and (4.13) we get

$$z_\lambda = \max\Big\{\frac{A}{y_{\lambda-1}}, \frac{A}{y_{\lambda-2}}, ..., \frac{A}{y_{\lambda-k-1}}\Big\} \geq \frac{A}{y_\mu} > \overline{z}$$

which contradicts to (4.13). So $\lambda = \mu$ and then $z_\lambda < \overline{z}$, $y_\lambda < \overline{y}$. Then from (4.2) we have

$$y_\lambda z_\lambda < A$$

and so (y_n, z_n) is eventually periodic of period $k+2$ if (4.10) holds.

Finally, suppose that

$$y_{k+2}z_{k+2} = A. \tag{4.14}$$

From (4.4) it obvious that

$$y_{k+2} \geq \frac{A}{z_i}, \quad z_{k+2} \geq \frac{A}{y_i}, \quad i = 1, 2, ..., k+1. \tag{4.15}$$

Therefore, relations (4.4), (4.14) and (4.15) imply that

$$\begin{aligned} y_{k+3} &= \max\Big\{y_{k+2}, \frac{A}{z_{k+1}}, ..., \frac{A}{z_2}\Big\} = y_{k+2} \\ z_{k+3} &= \max\Big\{z_{k+2}, \frac{A}{y_{k+1}}, ..., \frac{A}{y_2}\Big\} = z_{k+2}. \end{aligned} \tag{4.16}$$

Hence, using (4.16) and working inductively we can easily prove that

$$y_{k+i} = y_{k+2}, \quad z_{k+i} = z_{k+2}, \quad i = 3, 4, ...$$

and so it is obvious that (y_n, z_n) is eventually periodic of period $k+2$ if (4.14) holds. This completes the proof of the lemma.

Proposition 4.1 *Consider equation (1.1) where A is a positive real constant and $x_{-k}, x_{-k+1}, ..., x_0$ are positive fuzzy numbers. Then every positive solution of equation (1.1) is eventually periodic of period $k+2$.*

Proof Let x_n be a positive solution of (1.1) with initial values $x_{-k}, x_{-k+1}, ..., x_0$ such that (3.1) and (3.3) hold. From Proposition 3.1 $(L_{n,a}, R_{n,a})$, $n = 1, 2, ...$, $a \in (0, 1]$ satisfies system (3.2). Using Lemma 4.2 we have that

$$L_{n+k+2,a} = L_{n,a}, \quad R_{n+k+2,a} = R_{n,a}, \quad n = 2k+4, 2k+5, ..., a \in (0, 1]. \tag{4.17}$$

Therefore, from (3.3) and (4.17) we have that x_n is eventually periodic of period $k+2$. This completes the proof of the proposition.

5 PERIODICITY OF THE SOLUTIONS OF FUZZY EQUATION (1.2)

We study the periodicity of the positive solutions of (1.2). We need the following lemma.

Lemma 5.1 *Consider the system of difference equations*

$$y_{n+1} = \max\Big\{\frac{A_0}{z_n}, \frac{A_1}{z_{n-1}}\Big\}, \quad z_{n+1} = \max\Big\{\frac{A_0}{y_n}, \frac{A_1}{y_{n-1}}\Big\} \tag{5.1}$$

where A_0, A_1 are positive constants and the initial values y_{-1}, y_0, z_{-1}, z_0 are positive real numbers. Then the following statements are true:

(i) If

$$A_0 > A_1 \tag{5.2}$$

then every positive solution of system (5.1) is eventually periodic of period two.

(ii) If

$$A_0 = A_1 \tag{5.3}$$

then every positive solution of system (5.1) is eventually periodic of period three.

(iii) If

$$A_0 < A_1 \tag{5.4}$$

then every positive solution of system (5.1) is eventually periodic of period four.

Proof Let (y_n, z_n) be a positive solution of (5.1).

(i). Firstly, we prove that if there exists an $m \in \{1, 2, ...\}$ such that

$$A_1 \leq y_m z_m \leq \frac{A_0^2}{A_1} \tag{5.5}$$

then (y_n, z_n) is eventually periodic of period two. Relations (5.1) imply that

$$z_n y_{n-1} \geq A_0, \; y_n z_{n-1} \geq A_0, \; n = 1, 2, ... \tag{5.6}$$

From (5.5) and (5.6) we get

$$\frac{A_1}{z_{m-1}} \leq \frac{A_1}{A_0} y_m \leq \frac{A_0}{z_m}, \quad \frac{A_1}{y_{m-1}} \leq \frac{A_1}{A_0} z_m \leq \frac{A_0}{y_m}. \tag{5.7}$$

Using (5.1) and (5.7) it follows that

$$y_{m-1} = \max\left\{\frac{A_0}{z_m}, \frac{A_1}{z_{m-1}}\right\} = \frac{A_0}{z_m}, \quad z_{m+1} = \frac{A_0}{y_m}. \tag{5.8}$$

From (5.1), (5.5) and (5.8) we can easily prove that

$$y_{m+2} = \max\left\{y_m, \frac{A_1}{z_m}\right\} = y_m, \quad z_{m+2} = z_m. \tag{5.9}$$

Moreover, using (5.1), (5.5), (5.8) and (5.9) we get

$$y_{m+3} = \max\left\{\frac{A_0}{z_m}, \frac{A_1}{A_0} y_m\right\} = \frac{A_0}{z_m} = y_{m+1}, \quad z_{m+3} = z_{m+1}. \tag{5.10}$$

Therefore, using (5.1), (5.9), (5.10) and working inductively we can easily prove that

$$y_{n+2} = y_n, \; z_{n+2} = z_n, \; n = m+2, m+3, ...$$

and so (y_n, z_n) is eventually periodic of period two.

Now, we prove that there exists an $m \in \{1, 2, ...\}$ such that (5.5) holds. If there exists an $m \in \{1, 2\}$ such that (5.5) is satisfied then the proof is completed. Now, suppose that for any $m \in \{1, 2\}$ relation (5.5) is not true. We claim that there exists a $w \in \{1, 2, 3\}$ such that

$$u_w = y_w z_w < A_1. \tag{5.11}$$

If for $w = 1, 2$ relation (5.11) does not hold, then from (5.2) and since (5.5) is not true for $m = 1, 2$ we have

$$u_w > \frac{A_0^2}{A_1} > A_1, \quad w = 1, 2. \tag{5.12}$$

Hence, from (5.1), (5.6) and (5.12) we get

$$y_3 z_3 = \max\left\{\frac{A_0^2}{y_2 z_2}, \frac{A_0 A_1}{y_1 z_2}, \frac{A_0 A_1}{y_2 z_1}, \frac{A_1^2}{y_1 z_1}\right\} < A_1$$

and so our claim is true.

Now, we prove that if $u_n = y_n z_n$ then for $n = w, w+1, ...$

$$u_{n+1} = \max\left\{\frac{A_0^2}{u_n}, \frac{A_1^2}{u_{n-1}}\right\}. \tag{5.13}$$

From (5.1) we get

$$u_{n+1} = \max\left\{\frac{A_0^2}{u_n}, \frac{A_0 A_1}{y_n z_{n-1}}, \frac{A_0 A_1}{z_n y_{n-1}}, \frac{A_1^2}{u_{n-1}}\right\}. \tag{5.14}$$

Then from (5.2), (5.6), (5.11) and (5.14) we can easily prove that (5.13) is true for $n = w$. Furthermore, relations (5.1) imply that

$$z_{n+1} y_n = \max\left\{A_0, \frac{A_1 y_n}{y_{n-1}}\right\}, \quad y_{n+1} z_n = \max\left\{A_0, \frac{A_1 z_n}{z_{n-1}}\right\}. \tag{5.15}$$

Moreover, from (5.6) and (5.11) we have that

$$\frac{y_w}{y_{w-1}} = \frac{y_w z_w}{z_w y_{w-1}} < \frac{A_1}{A_0}, \quad \frac{z_w}{z_{w-1}} < \frac{A_1}{A_0}. \tag{5.16}$$

Therefore, from (5.2), (5.15) and (5.16) it follows that

$$z_{w+1} y_w = y_{w+1} z_w = A_0. \tag{5.17}$$

Then using (5.15) and (5.17) we have

$$z_{w+2} y_{w+1} = y_{w+2} z_{w+1}. \tag{5.18}$$

Hence, using (5.15), (5.17), (5.18) and working inductively we take

$$z_n y_{n+1} = y_n z_{n+1}, \quad n = w, w+1, ... \tag{5.19}$$

So, from (5.14) and (5.19) we can easily prove that (5.13) is true.

From (5.2) and (5.11) there exists an $r \in \{0, 1, ...\}$ such that

$$\Big(\frac{A_1}{A_0}\Big)^{2r+2} \leq \frac{u_w}{A_1} \leq \Big(\frac{A_1}{A_0}\Big)^{2r}. \tag{5.20}$$

Now, we prove that for all $s = 0, 1, ..., r+1$

$$u_{w+3s} = \frac{u_w A_0^{2s}}{A_1^{2s}}, \quad u_{w+3s+1} = \frac{A_1^{2s}}{u_w A_0^{2s-2}}. \tag{5.21}$$

In view of (5.2), (5.6) and (5.11) we get

$$\frac{A_1^2}{u_{w-1}} < \frac{A_0^2}{u_{w-1}} \leq u_w < \frac{A_1^2}{u_w} < \frac{A_0^2}{u_w}. \tag{5.22}$$

From (5.13) and (5.22) we take

$$u_{w+1} = \max\Big\{\frac{A_0^2}{u_w}, \frac{A_1^2}{u_{w-1}}\Big\} = \frac{A_0^2}{u_w}. \tag{5.23}$$

Therefore, relation (5.23) implies that (5.21) is true for $s = 0$. Suppose that (5.21) is true for an $s = j \in \{0, 1, ..., r\}$. Then from (5.2), (5.13) for $n = w + 3j + 1$, (5.20) and (5.21) for $s = j$ we have

$$u_{w+3j+2} = \max\Big\{\frac{A_0^{2j} u_w}{A_1^{2j}}, \frac{A_1^{2j+2}}{u_w A_0^{2j}}\Big\} = \frac{A_1^{2j+2}}{u_w A_0^{2j}}. \tag{5.24}$$

Moreover, using (5.2), (5.13) for $n = w + 3j + 2$, (5.21) for $s = j$ and (5.24) it follows that

$$u_{w+3j+3} = \max\Big\{\frac{A_0^{2j+2} u_w}{A_1^{2j+2}}, \frac{A_0^{2j-2} u_w}{A_1^{2j-2}}\Big\} = \frac{A_0^{2j+2} u_w}{A_1^{2j+2}}. \tag{5.25}$$

Finally, from relations (5.2), (5.13) for $n = w + 3j + 3$, (5.20), (5.24) and (5.25) we take

$$u_{w+3j+4} = \max\Big\{\frac{A_1^{2j+2}}{A_0^{2j} u_w}, \frac{A_0^{2j} u_w}{A_1^{2j}}\Big\} = \frac{A_1^{2j+2}}{A_0^{2j} u_w}. \tag{5.26}$$

Therefore, (5.21) is true for $s = 0, 1, ..., r+1$. Then from (5.20) and (5.26) for $j = r$ it follows that

$$A_1 \leq u_{w+3r+4} \leq \frac{A_0^2}{A_1}$$

which implies that if $m = w + 3r + 4$ then (5.5) holds. Therefore, (y_n, z_n) is eventually periodic of period two. This completes the proof of Statement (i).

(ii) From Lemma 4.2 the proof of Statement (ii) follows immediately.

(iii) Firstly, we prove that if there exists an $m \in \{1, 2, ...\}$ such that

$$\frac{A_0^2}{A_1} \leq y_m z_m \leq A_1 \tag{5.27}$$

then (y_n, z_n) is eventually periodic of period four. From (5.6) and (5.27) we get

$$\frac{A_1}{z_{m-1}} \leq \frac{A_1}{A_0} y_m, \quad \frac{A_0}{z_m} \leq \frac{A_1}{A_0} y_m, \quad \frac{A_1}{y_{m-1}} \leq \frac{A_1}{A_0} z_m, \quad \frac{A_0}{y_m} \leq \frac{A_1}{A_0} z_m. \tag{5.28}$$

So, in view of (5.1) and (5.28) it is obvious that

$$y_{m+1} = \max\left\{\frac{A_0}{z_m}, \frac{A_1}{z_{m-1}}\right\} \leq \frac{A_1}{A_0} y_m, \quad z_{m+1} \leq \frac{A_1}{A_0} z_m. \tag{5.29}$$

From (5.6) and (5.27) we get

$$\frac{A_0}{z_{m+1}} \leq y_m \leq \frac{A_1}{z_m}, \quad \frac{A_0}{y_{m+1}} \leq z_m \leq \frac{A_1}{y_m}$$

and so from (5.1) we have

$$y_{m+2} = \max\left\{\frac{A_0}{z_{m+1}}, \frac{A_1}{z_m}\right\} = \frac{A_1}{z_m}, \quad z_{m+2} = \frac{A_1}{y_m}. \tag{5.30}$$

Using (5.27) and (5.29) it follows that

$$\frac{A_0}{A_1} y_m \leq \frac{A_0}{z_m} \leq \frac{A_1}{z_{m+1}}, \quad \frac{A_0}{A_1} z_m \leq \frac{A_0}{y_m} \leq \frac{A_1}{y_{m+1}}$$

and hence, from (5.1) and (5.30) it follows that

$$y_{m+3} = \max\left\{\frac{A_0 y_m}{A_1}, \frac{A_1}{z_{m+1}}\right\} = \frac{A_1}{z_{m+1}}, \quad z_{m+3} = \frac{A_1}{y_{m+1}}. \tag{5.31}$$

In view of (5.1), (5.29), (5.30) and (5.31) we get

$$y_{m+4} = \max\left\{\frac{A_0 y_{m+1}}{A_1}, y_m\right\} = y_m, \quad z_{m+4} = z_m. \tag{5.32}$$

In addition, from (5.1), (5.6), (5.31) and (5.32) we have

$$y_{m+5} = \max\left\{\frac{A_0}{z_m}, y_{m+1}\right\} = y_{m+1}, \quad z_{m+5} = z_{m+1}. \tag{5.33}$$

Therefore, using (5.1), (5.32), (5.33) and working inductively we can easily prove that

$$y_{n+4} = y_n, \ z_{n+4} = z_n, \ n = m+2, m+3, ...$$

and so (y_n, z_n) is eventually periodic of period four.

Now, we prove that there exists an $m \in \{1, 2, ...\}$ such that (5.27) holds. If there exists an $m \in \{1, 2, 3\}$ such that (5.27) is satisfied then the proof is completed. Now, suppose that for any $m \in \{1, 2, 3\}$ relation (5.27) is not true. We claim that there exists a $p \in \{1, 2, 3\}$ such that

$$u_p = y_p z_p < \frac{A_0^2}{A_1}. \tag{5.34}$$

If for $p = 1, 2$ relation (5.34) does not hold, then since (5.27) is not true for $m = 1, 2$ we have

$$u_p > A_1, \ p = 1, 2. \tag{5.35}$$

Hence, from (5.1), (5.6), (5.35) and since (5.27) is not satisfied for $p \in \{1, 2, 3\}$ we get

$$y_3 z_3 = \max\left\{\frac{A_0^2}{y_2 z_2}, \frac{A_0 A_1}{y_1 z_2}, \frac{A_0 A_1}{y_2 z_1}, \frac{A_1^2}{y_1 z_1}\right\} < \frac{A_0^2}{A_1}$$

and so our claim is true.

In addition, we prove that for $n = p, p+1, ...$ relation (5.13) is satisfied. From (5.6), (5.14) and (5.34) we have that (5.13) is true for $n = p$. Furthermore, since from (5.6) and (5.34) the following relations

$$\frac{y_p}{y_{p-1}} = \frac{y_p z_p}{z_p y_{p-1}} < \frac{A_0}{A_1}, \quad \frac{z_p}{z_{p-1}} < \frac{A_0}{A_1}$$

are satisfied, then using (5.15) we obtain that

$$z_{p+1} y_p = y_{p+1} z_p = A_0.$$

Then arguing as in Statement (i) we can easily prove that (5.13) is true for $n = p+1, p+2, ...$

From (5.4) and (5.34) there exists a $q \in \{0, 1, ...\}$ such that

$$\left(\frac{A_0}{A_1}\right)^{2q+2} \leq \frac{u_p A_1}{A_0^2} \leq \left(\frac{A_0}{A_1}\right)^{2q}. \tag{5.36}$$

Now, we prove that for all $s = 0, 1, ..., q+1$

$$u_{p+3s} = \frac{u_p A_1^{2s}}{A_0^{2s}}, \quad u_{p+3s+1} = \frac{A_0^{2s+2}}{u_p A_1^{2s}}. \tag{5.37}$$

From (5.6) and (5.34) we get

$$\frac{A_1^2}{u_{p-1}} \leq \frac{A_1^2 u_p}{A_0^2} < \frac{A_0^2}{u_p}. \tag{5.38}$$

From (5.13) for $n = p$ and (5.38) we take

$$u_{p+1} = \max\left\{\frac{A_0^2}{u_p}, \frac{A_1^2}{u_{p-1}}\right\} = \frac{A_0^2}{u_p}. \tag{5.39}$$

Therefore, from (5.39) relation (5.37) is true for $s = 0$. Suppose that (5.37) is true for an $s = j \in \{0, 1, ..., q\}$. Then from (5.4), (5.13) for $n = p+3j+1$, (5.36) and (5.37) for $s = j$ we have

$$u_{p+3j+2} = \max\left\{\frac{A_1^{2j}u_p}{A_0^{2j}}, \frac{A_0^{2j}}{u_pA_1^{2j-2}}\right\} = \frac{A_0^{2j}}{u_pA_1^{2j-2}}. \tag{5.40}$$

In addition, using (5.4), (5.13) for $n = p+3j+2$, (5.37) for $s = j$ and (5.40) it follows that

$$u_{p+3j+3} = \max\left\{\frac{A_1^{2j-2}u_p}{A_0^{2j-2}}, \frac{A_1^{2j+2}u_p}{A_0^{2j+2}}\right\} = \frac{A_1^{2j+2}u_p}{A_0^{2j+2}}. \tag{5.41}$$

Finally, from relations (5.4), (5.13) for $n = p + 3j + 3$, (5.36), (5.40) and (5.41) we get

$$u_{p+3j+4} = \max\left\{\frac{A_0^{2j+4}}{A_1^{2j+2}u_p}, \frac{A_1^{2j}u_p}{A_0^{2j}}\right\} = \frac{A_0^{2j+4}}{A_1^{2j+2}u_p}. \tag{5.42}$$

Therefore, (5.37) is true for $s = 0, 1, ..., q+1$. Hence, from (5.36) and (5.42) for $j = q$ it follows that

$$\frac{A_0^2}{A_1} \leq u_{p+3q+4} \leq A_1$$

which implies that if $m = p + 3q + 4$ then (5.27) holds. Therefore, (y_n, z_n) is eventually periodic of period four. This completes the proof of Statement (iii). Thus, the proof of the lemma is completed.

Proposition 5.1 *Consider the fuzzy difference equation (1.2) where A_i, $i = 0, 1$ are positive real constants and the initial values x_i, $i = -1, 0$ are positive fuzzy numbers. Then if relation (5.2) (resp. (5.3)) (resp. (5.4)) holds, every positive solution of equation (1.2) is eventually periodic of period two (resp. three) (resp. four).*

Proof Let x_n be a positive solution of (1.2) with initial values x_i, $i = -1, 0$ such that relations (3.3), (3.5) hold, then $(L_{n,a}, R_{n,a})$, $n = 1, 2, ..., a \in (0, 1]$ satisfies system (3.6).

Firstly, suppose that (5.2) is satisfied. We define the set $E \subset (0, 1]$ as follows:
For any $a \in E$ there exists an $m_a \in \{1, 2\}$ such that

$$A_1 \leq u_{m_a,a} \leq \frac{A_0^2}{A_1}, \quad u_{n,a} = L_{n,a}R_{n,a}, \ n = 1, 2, ..., \ a \in E. \tag{5.43}$$

Then from Statement (i) of Lemma 5.1 the sequences $L_{n,a}, R_{n,a}$, $a \in E$ are periodic sequences of period two for $n \geq 2$. Moreover, since for any $a \in (0,1] - E$ the relation (5.43) does not hold, then from Statement (i) of Lemma 5.1 for any $a \in (0,1] - E$ there exists a $w_a \in \{1,2,3\}$ and an $r_a \in \{0,1,...\}$ such that analogous relations (5.11), (5.20) and (5.21) are satisfied and so form Statement (i) of Lemma 5.1 $L_{n,a}, R_{n,a}$, $a \in (0,1]-E$ are periodic sequences of period two for $n \geq w_a + 3r_a + 4$ and so for $n \geq 3r_a + 7$.

We prove that there exists an $r \in \{0,1,...\}$ such that

$$r+1 \geq r_a, \quad a \in (0,1] - E. \tag{5.44}$$

Since x_i, $i = 1,2,3$ are positive fuzzy numbers there exist positive real numbers K, L such that $[L_{i,a}, R_{i,a}] \subset [K, L]$, $i = 1,2,3$, $\alpha \in (0,1] - E$. Then from analogous relation (5.20) there exists an $r \in \{0,1,...\}$ such that

$$\left(\frac{A_1}{A_0}\right)^{2r+2} \leq \frac{K^2}{A_1} \leq \frac{u_{w_a,a}}{A_1} \leq \left(\frac{A_1}{A_0}\right)^{2r_a}, \ a \in (0,1] - E, \ n = 1,2,3$$

and so from (5.2) relation (5.44) is satisfied. Therefore, from (5.44) it follows that $L_{n,a}, R_{n,a}$, $a \in (0,1] - E$ are periodic sequences of period two for $n \geq 3r + 10$ and so x_n is eventually periodic of period two.

If (5.3) holds then from Proposition 4.1 we have that every positive solution is eventually periodic of period three.

Arguing as above and using Statement (iii) of Lemma 5.1 we can easily prove that every positive solution of (1.2) is eventually periodic of period four if relation (5.4) holds. This completes the proof of the proposition.

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